Some Properties of Three Coupled Waves*

LASZLO SOLYMAR†

Summary—The paper deals with the problem of three waves, 1, 2, and 3, in which waves 2 and 3 are coupled to wave 1 but not to each other. The general solution for the amplitudes of the waves is given in closed form. It is shown that for certain values of the parameters growing waves can exist. Numerical solutions for the location of the boundaries of the growing wave regions are plotted. It is shown furthermore that under certain conditions the power can be completely transferred from wave 1 to waves 2 and 3.

Examples on traveling-wave tubes, waveguide couplers, and backward-wave oscillators illustrate the applicability of the theory.

I. Introduction

Coupled wave theory[1–3] has in the recent past proved to be a powerful approach to the approximate solution of a wide variety of problems. Not only are the results often quantitatively of sufficient accuracy, but the physical picture which emerges is also of great value in understanding the essential nature of the particular problem.

This paper is an attempt to extend the quantitative treatment to three lossless coupled waves. In many cases it turns out that two of the three waves are uncoupled. Accordingly, this restriction has been imposed in the paper, with a considerable saving in complexity. This picture can be successfully applied to the description of waveguide couplers, traveling-wave tubes, and backward-wave oscillators, but naturally the conclusions are much more general and are valid for any coupled system.

In Section II the general solution of the coupled wave differential equation system is given in closed form. In Section III the condition for growing waves is found and the results are plotted in Figs. 1–8. In Section IV conditions of complete power transfer are investigated. In Section V four examples are given which demonstrate the applicability of the general formulas derived.

II. The Solution of the Coupled Wave Differential Equation System

The generality of the solution will be restricted in the following aspects:

1) Wave 2 and wave 3 are not coupled.
2) The couplings between waves 1 and 2, and waves 1 and 3 are assumed to be uniform, *i.e.*, they are independent of the space variable z.
3) The phase velocities of all three waves are in the positive direction of the z axis.
4) At the beginning of the coupled system all the power is in wave 1.

Subject to the above restrictions the coupled wave differential equation system can be written as follows:[4]

$$-\frac{dE_1}{dz} = j\beta_1 E_1 + jd_{12}E_2 + jd_{13}E_3$$

$$-\frac{dE_2}{dz} = jf_{12}d_{12}E_1 + j\beta_2 E_2$$

$$-\frac{dE_3}{dz} = jf_{13}d_{13}E_1 + j\beta_3 E_3 \qquad (1)$$

where

E_1, E_2, E_3 = the amplitudes of waves 1, 2, and 3 respectively,
$\beta_1, \beta_2, \beta_3$ = the propagation coefficients of waves 1, 2, and 3 respectively,
d_{12}, d_{13} = coupling coefficients between waves 1 and 2, and 1 and 3 respectively, and
f_{12}, f_{13} = ± 1 if the energy velocity of wave 2, 3 is in the same/opposite direction as that of wave 1.

We now solve the differential equation system by assuming the following form for the amplitudes:

$$E_1 = A_1 e^{jt_1 z} + A_2 e^{jt_2 z} + A_3 e^{jt_3 z}$$

$$E_2 = B_1 e^{jt_1 z} + B_2 e^{jt_2 z} + B_3 e^{jt_3 z}$$

$$E_3 = C_1 e^{jt_1 z} + C_2 e^{jt_2 z} + C_3 e^{jt_3 z}. \qquad (2)$$

Substituting (2) into (1) the unknown coefficients can be determined, while t_1, t_2, t_3 are the roots of the following third power equation:

$$t^3 + T_1 t^2 + T_2 t + T_3 = 0 \qquad (3)$$

where

$$T_1 = \beta_1 + \beta_2 + \beta_3$$

$$T_2 = \beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3 - f_{12}d_{12}^2 - f_{13}d_{13}^2$$

$$T_3 = \beta_1\beta_2\beta_3 - f_{12}d_{12}^2\beta_3 - f_{13}d_{13}^2\beta_2. \qquad (4)$$

Applying furthermore the boundary conditions, in accordance with restriction 4),

$$E_1(0) = 1, \qquad E_2(0) = 0, \qquad E_3(0) = 0. \qquad (5)$$

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† Standard Telecommunications Labs., Ltd., Harlow, Eng.

[1] J. R. Pierce, "Coupling of modes of propagation," *J. Appl. Phys.*, vol. 25, pp. 179–183; February, 1954.

[2] S. E. Miller, "Coupled wave theory and waveguide applications," *Bell. Sys. Tech., J.*, vol. 33, pp. 661–719; May, 1954.

[3] J. R. Pierce, "The wave picture of microwave tubes," *Bell Sys. Tech. J.*, vol. 33, pp. 1343–1372; November, 1954.

[4] The relations between the matrix elements are a direct consequence of the conservation of energy.

The general solution can be written in the following closed form:

$$E_1(z) = \sum_{i=1}^{3} \frac{(t_i + \beta_2)(t_i + \beta_3)}{(t_i - t_{i+1})(t_i - t_{i-1})} e^{jt_iz}$$

$$E_2(z) = -f_{12}d_{12} \sum_{i=1}^{3} \frac{t_i + \beta_3}{(t_i - t_{i+1})(t_i - t_{i-1})} e^{jt_iz}$$

$$E_3(z) = -f_{13}d_{13} \sum_{i=1}^{3} \frac{t_i + \beta_2}{(t_i - t_{i+1})(t_i - t_{i-1})} e^{jt_iz} \quad (6)$$

where

$$t_0 = t_3 \quad \text{and} \quad t_4 = t_1.$$

III. The Condition for the Existence of Complex Roots

The solution of (3) may result in three real roots or in one real and two complex roots. It may be seen from (6) that complex roots mean an attenuating and a growing wave. It should be appreciated, however, that the existence of a growing wave solution does not necessarily imply "amplification" in the usual sense. This will always depend on the boundary conditions imposed by the physics of the problem. In fact, amplification can take place even when all the roots are purely imaginary, two examples being the backward-wave amplifier and the crestatron. Nevertheless, in most physical problems, the demarcation between the regions of pure imaginary and complex roots is of fundamental significance.

By introducing the new variable

$$u = t + \tfrac{1}{3}T_1 \quad (7)$$

we bring (3) into the following more appropriate form:

$$u^3 + 3Hu + G = 0 \quad (8)$$

where

$$3H = T_2 - \tfrac{1}{3}T_1^2 = -\tfrac{1}{6}[r^2 + p^2 + (r-p)^2 + 6(f_{12}d_{12}^2 + f_{13}d_{13}^2)] \quad (9)$$

$$G = \frac{2}{27}T_1^3 - \tfrac{1}{3}T_1T_2 + T_3 = -\frac{1}{27}[(p+r)(2p-r) \cdot (2r-p) + 9(2p-r)f_{12}d_{12}^2 + (2r-p)f_{13}d_{13}^2] \quad (10)$$

$$p = \beta_3 - \beta_1, \qquad r = \beta_2 - \beta_1. \quad (11)$$

It can be seen that both H and G depend only on the difference of the propagation coefficients. This is physically obvious, because it is always possible to regard one of the waves as stationary.

Now we can express in mathematical form the condition for complex roots. Eq. (3) has two complex roots, if

$$M = G^2 + 4H^3 > 0. \quad (12)$$

Substituting (9) and (10) into (12) and arranging by powers of p, we obtain:

$$\begin{aligned} M = &-p^4(r^2 + 4f_{12}d_{12}^2) + 2p^3r(r^2 - f_{13}d_{13}^2 + 4f_{12}d_{12}^2) \\ &- p^2[r^4 + 2r^2(f_{12}d_{12}^2 + f_{13}d_{13}^2) - 8d_{12}^4 \\ &\qquad + 20f_{12}f_{13}d_{12}^2d_{13}^2 + d_{13}^4] \\ &+ 2pr[r^2(4f_{13}d_{13}^2 - f_{12}d_{12}^2) - 4d_{12}^4 \\ &\qquad + 19f_{12}f_{13}d_{12}^2d_{13}^2 - 4d_{13}^4] \\ &- 4r^4f_{13}d_{13}^2 - r^2(d_{12}^4 + 20f_{12}f_{13}d_{12}^2d_{13}^2 - 8d_{13}^4) \\ &- 4(f_{12}d_{12}^2 + f_{13}d_{13}^2)^3. \end{aligned} \quad (13)$$

Since M depends on the direction of the energy velocities, v_{e1}, v_{e2}, v_{e3}, we have to investigate three cases. Denoting an energy velocity in the same direction as wave 1 by s, and in the opposite direction as wave 1 by o, we have the following three cases:

	(a)	(b)	(c)
v_{e2}	s	s	o
v_{e3}	s	o	o.

The fourth possibility (os) has been omitted as we are not distinguishing between waves 2 and 3. It can be shown from (13) that case (a) always leads to $M<0$ so that here no growing wave solution exists.

The study of case (b) reveals (Figs. 1–5) that for certain values of p, r, d_{12}, d_{13} growing wave solution exists. The figures show the $M=0$ lines on the p/d_{12}, r/d_{12} plane for different values of d_{13}^2/d_{12}^2. The curves are plotted only for positive values of p/d_{12}, because of the relation $M(p/d_{12}, r/d_{12}) = M(-p/d_{12}, -r/d_{12})$. Each of these Figures can be roughly divided into three parts:

1) The neighborhood of the origin,
2) the neighborhood of the r/d_{12} axis (except near the origin), and
3) the neighborhood of the $p=r$ line (except near the origin).

The following conclusions can be drawn for each, respectively:

1) The greater the coupling to wave 3 compared with that to wave 2, the greater is the extent of the growing wave region near the origin.
2) If $p=\beta_3-\beta_1\approx 0$, *i.e.*, the velocity of wave 3 is near to that of wave 1, growing wave solution always exists irrespective of the value of r/d_{12}. As r/d_{12} goes to $(\pm\infty)$ the $M=0$ line approaches the $p/d_{12}=2d_{13}/d_{12}$ asymptote (broken lines).
3) If $p>r>0$, but p and r are nearly equal, growing wave solution exists. This means physically, that if wave 3 is slower than wave 2 (both being slower than wave 1), but the velocity difference between waves 2 and 3 is sufficiently small, growing wave solution exists irrespective of the velocity of wave 1.

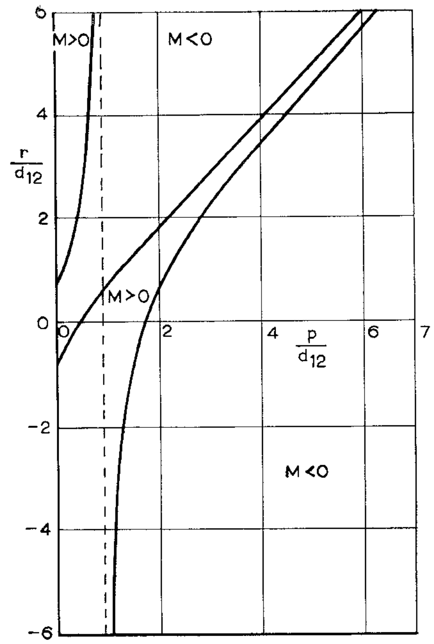

Fig. 1—The domain of complex roots for $f_{12}=1, f_{13}=-1, d_{13}^2/d_{12}^2=1/5$.

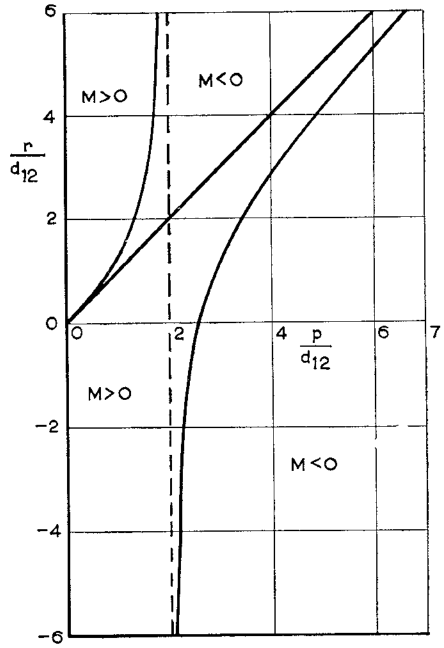

Fig. 3—The domain of complex roots for $f_{12}=1, f_{13}=-1, d_{13}^2/d_{12}^2=1$.

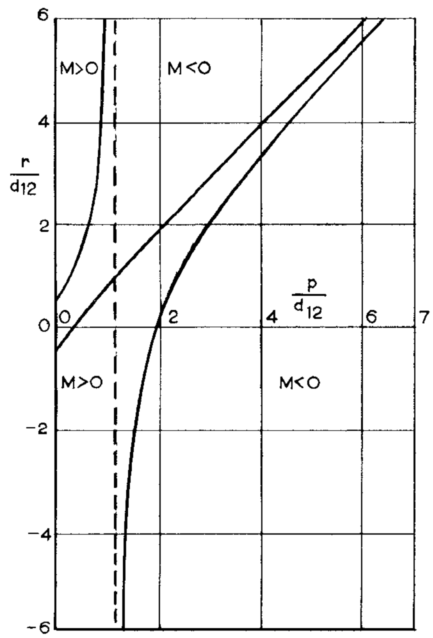

Fig. 2—The domain of complex roots for $f_{12}=1, f_{13}=-1, d_{13}^2/d_{12}^2=1/3$.

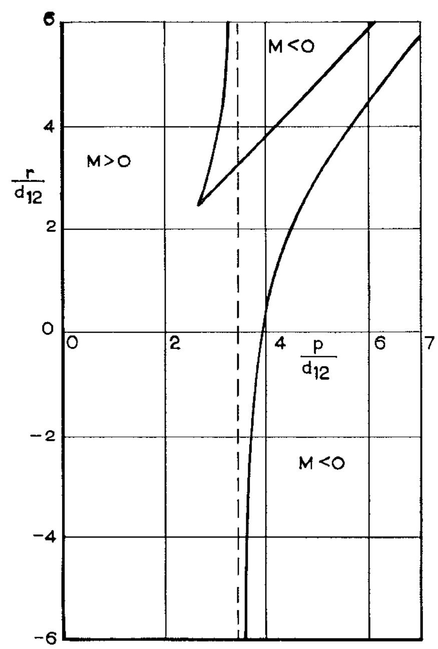

Fig. 4—The domain of complex roots for $f_{12}=1, f_{13}=1, d_{13}^2/d_{12}^2=3$.

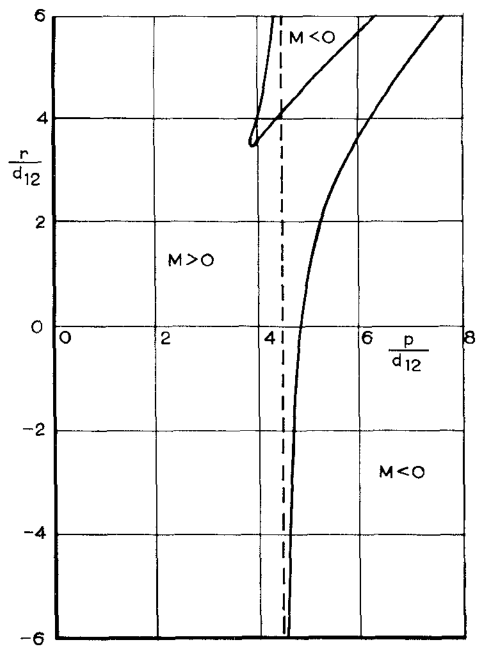

Fig. 5—The domain of complex roots for $f_{12}=1$, $f_{13}=-1$, $d_{13}^2/d_{12}^2=5$.

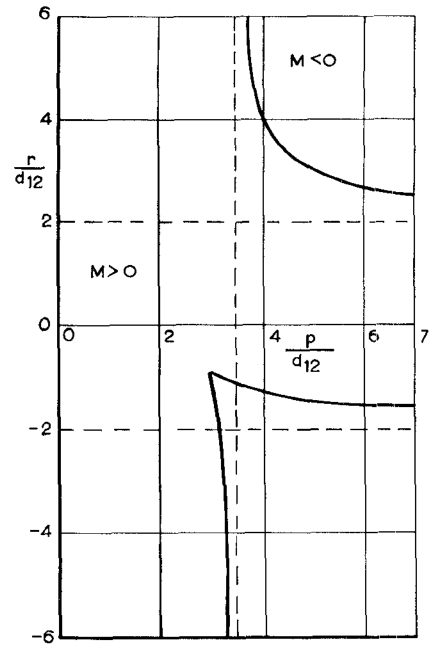

Fig. 7—The domain of complex roots for $f_{12}=-1$, $f_{13}=-1$, $d_{13}^2/d_{12}^2=3$.

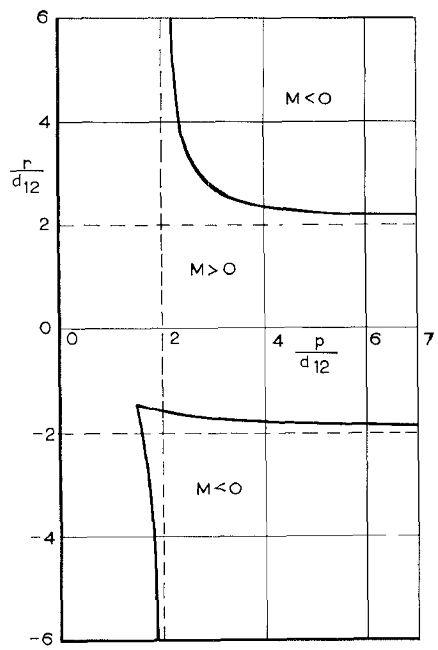

Fig. 6—The domain of complex roots for $f_{12}=-1$, $f_{13}=-1$, $d_{13}^2/d_{12}^2=1$.

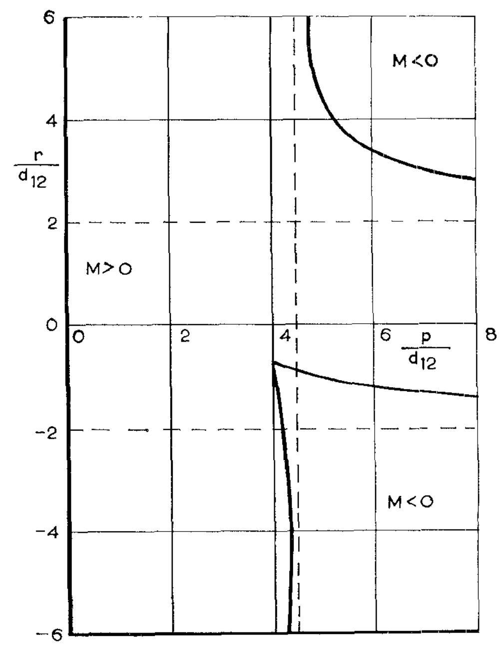

Fig. 8—The domain of complex roots for $f_{12}=-1$, $f_{13}=-1$, $d_{13}^2/d_{12}^2=5$.

The $M=0$ lines for case (c) are plotted in Figs. 6–8 for $d_{13}^2/d_{12}^2=1$, 3, 5. It may be seen that growing wave solution exists, if the velocity of either of the waves 2 or 3 is near to that of wave 1. The maximum velocity difference between wave 1 and 2, 3 which still leads to complex roots increases as the coupling between 1 and 2, 3 increases. The asymptotes are:

$$\frac{p}{d_{12}} \to \infty \qquad \frac{r}{d_{12}} = \pm 2$$

$$\frac{r}{d_{12}} \to \infty \qquad \frac{p}{d_{12}} = 2\frac{d_{13}}{d_{12}} \cdot$$

IV. Conditions of Complete Power Transfer

According to our boundary conditions, all the power is contained in wave 1 at $z=0$. In this section we shall investigate under what conditions this power can be completely transferred to waves 2 and 3.

Mathematically, it is equivalent to find the parameters which give $E_1(z)=0$. Since it does not appear that the general solution can be expressed in closed analytical form, we restrict generality and give only three solutions.

1) Complete power transfer is possible, if all three waves have the same velocity. Using the condition $\beta_1=\beta_2=\beta_3$, the amplitudes of the waves can be obtained from (6). Performing the calculations we get

$$E_1 = e^{-j\beta_1 z}\cos u_r z$$

$$E_2 = -jf_{12}d_{12}e^{-j\beta_1 z}\frac{\sin u_r z}{u_r}$$

$$E_3 = -jf_{13}d_{13}e^{-j\beta_1 z}\frac{\sin u_r z}{u_r} \tag{14}$$

where

$$u_r^2 = f_{12}d_{12}^2 + f_{13}d_{13}^2.$$

Thus complete power transfer takes place at the distance L, if

$$u_r^2 > 0 \text{ and } u_rL = (2k+1)\frac{\pi}{2}[k = 0, \pm 1, \pm 2 \cdots]. \tag{15}$$

2) Complete power transfer is possible, if the coupling coefficients are identical, the energy velocities of waves 2 and 3 are in the same direction, and the propagation coefficient of wave 1 is the arithmetical mean of the propagation coefficients of wave 2 and wave 3.

Using the conditions

$$\beta_1 = \frac{\beta_2+\beta_3}{2} \quad \text{and} \quad f_{12}d_{12} = f_{13}d_{13}, \tag{16}$$

the amplitudes can be written as follows:

$$E_1 = \frac{\exp(-j\beta_1 z)}{u_r^2}\{r^2 + 2f_{12}d_{12}^2\cos u_r z\}$$

$$E_2 = \exp(-j\beta_1 z)\frac{f_{12}d_{12}}{u_r^2}\cdot\{r(1-\cos u_r z) + ju_r\sin u_r z\}$$

$$E_3 = \exp(-j\beta_1 z)\frac{f_{13}d_{13}}{u_r^2}\cdot\{-r(1-\cos u_r z) + ju_r\sin u_r z\} \tag{17}$$

where

$$u_r^2 = r^2 + 2f_{12}d_{12}^2.$$

Complete power transfer takes place at the distance L, if

$$u_r^2 > 0, \qquad r^2 < 2d_{12}^2, \qquad \cos u_rL = -\frac{r^2}{2f_{12}d_{12}^2} \cdot \tag{18}$$

3) Complete power transfer is still possible, when neither the velocities nor the coupling coefficients are identical, but then rather strict relationships apply between the quantities p, r, d_{12}, and d_{13}.

The mathematical conditions are as follows:

$$d_{12}^2 = \frac{1}{9f_{12}}\frac{p}{p-r}(2r-p)^2$$

$$d_{13}^2 = \frac{1}{9f_{13}}\frac{r}{r-p}(2p-r)^2. \tag{19}$$

Subject to the above conditions, the amplitudes of the waves can be written as follows:

$$E_1 = \tfrac{1}{2}\exp\left(-j\frac{\beta_1+\beta_2+\beta_3}{3}z\right)\cdot\left\{1+\cos u_r z + j\tfrac{2}{3}(p+r)\frac{\sin u_r z}{u_r}\right\}$$

$$E_2 = \exp\left(-j\frac{\beta_1+\beta_2+\beta_3}{3}z\right)\frac{f_{12}d_{12}}{3u_r^2}\cdot\{(2p-r)(1-\cos u_r z) - 3ju_r\sin u_r z\}$$

$$E_3 = \exp\left(-j\frac{\beta_1+\beta_2+\beta_3}{3}z\right)\frac{f_{13}d_{13}}{3u_r^2}\cdot\{(2r-p)(1-\cos u_r z) - 3ju_r\sin u_r z\} \tag{20}$$

where

$$u_r^2 = \frac{2}{9}(2r-p)(r-2p).$$

Complete power transfer takes place at the distance L, if

$$u_r^2 > 0$$

and

$$u_rL = (2k+1)\pi[k = 0, \pm 1, \pm 2 \cdots]. \tag{21}$$

To see more clearly the relationship between the parameters, $f_{12}d_{12}^2/r^2$ and $f_{13}d_{13}^2/r^2$ are plotted in Fig. 9 against $a = p/r$. We can distinguish four regions.

1) $a<0, f_{12}=+1, f_{13}=+1$. All the waves in the same direction.
2) $0<a<\frac{1}{2}, f_{12}=-1, f_{13}=+1$. Wave 2 in the opposite direction.
3) $\frac{1}{2}<a<2$. No solution because $u_r^2<0$.
4) $a>2$, $f_{12}=1$, $f_{13}=-1$. Wave 3 in the opposite direction.

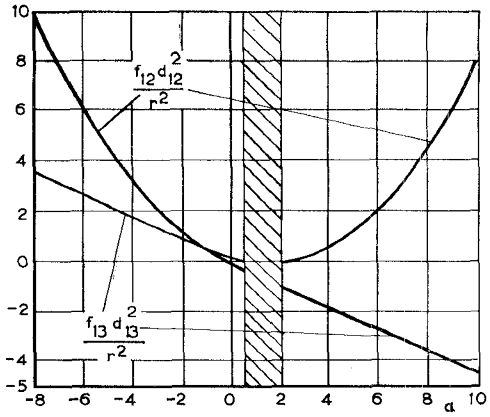

Fig. 9—A relationship between the parameters which results in complete power transfer.

V. Examples

The Amplification Domain of a Traveling-Wave Tube

In the case of a traveling-wave tube, the circuit wave is coupled to both the slow and the fast space charge waves. Accordingly, we can identify wave 1 with the circuit wave, wave 2 with the fast wave, and wave 3 with the slow wave. Writing the propagation and coupling coefficients into the usual notations of traveling-wave theory[5,6] we obtain

$$\beta_1 = \beta; \quad \beta_2 = \beta_e(1 - 2C\sqrt{QC}); \quad \beta_3 = \beta_e(1 + 2C\sqrt{QC})$$

$$d_{12} = d_{13} = \frac{\beta_e C}{2\sqrt[4]{QC}} \tag{22}$$

so that

$$\frac{p}{d_{12}} = -2b(QC)^{1/4} + 4(QC)^{3/4}$$

$$\frac{r}{d_{12}} = -2b(QC)^{1/4} - 4(QC)^{3/4}. \tag{23}$$

Since the coupling is the same to both space charge waves, the condition of amplification can be determined from Fig. 3. As p is always larger than r, the physically possible cases are below the $p/d_{12} = r/d_{12}$ line.

It may be seen from (23) that p/d_{12} and r/d_{12} are the functions of b and QC only. Therefore, the b = constant and QC = constant curves are plotted in Fig. 10, where for convenience the axes are rotated by 45 degrees. From the intersections with the $M=0$ lines the limiting values of b and QC can be determined. It may be seen that with decreasing values of QC the range of amplification is increased and pushed in the direction of lower values of b. The results of the two coupled waves theory[7] (taking account only of the circuit wave and the slow wave) are represented by the asymptotes (broken lines). If $QC>0.25$ the intersections with the asymptotes give good approximation.

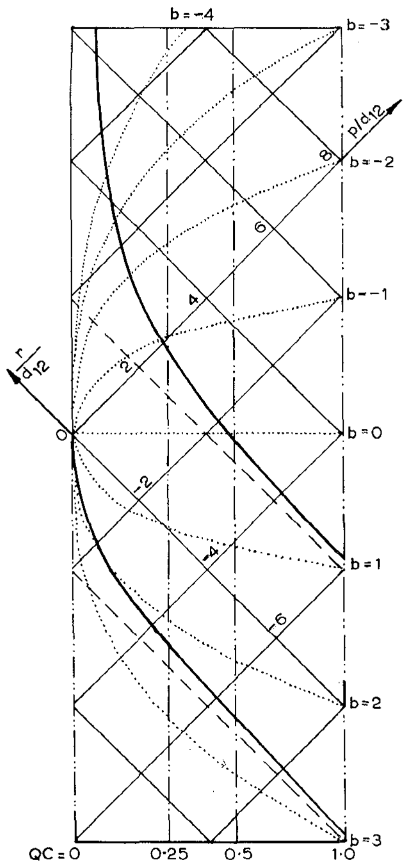

Fig. 10—The amplification domain of a traveling-wave tube. · · · b = constant curves. · · — · · — QC = constant curves.

The above results are, of course, familiar aspects of Pierce's theory of the traveling-wave tube;[5] the method of presenting the results here adopted is the "natural" one for the coupled wave picture, and shows up the essential physical phenomena from a different angle.

[5] J. R. Pierce, "Travelling-Wave Tubes," D. Van Nostrand Co., Inc., New York, N. Y.; 1950.
[6] R. W. Gould, "Traveling-wave couplers for longitudinal beam-type amplifiers," Proc. IRE, vol. 47, pp. 419–429; March, 1959.
[7] R. W. Gould, "A coupled mode description of the backward wave oscillator and the Kompfner dip condition," IRE Trans. on Electron Devices, vol. ED-2, pp. 37–42; October, 1955.

Traveling-Wave Tube at Kompfner Dip

For certain values of beam voltage and current, the power of the circuit wave can be completely transferred to the space charge waves.[8] This is known as the Kompfner dip condition and it is useful because it permits a direct measurement of the traveling-wave tube parameters. A number of authors[6,9,10] have published numerical solutions for the location of the point.

Unfortunately, our formulas derived in Section IV are not generally applicable owing to the severe restrictions represented by (19). It turns out that its validity is restricted to one particular case, namely to

$$b = -\frac{3}{2} \quad \text{and} \quad QC = \frac{5}{16}.$$

The amplitudes of the three waves then can be obtained from (20). Since this is the only analytical solution found so far for the Kompfner dip condition, it seems to be worthwhile to write up the formulas.

$$|E_1|^2 = \tfrac{1}{4}\{(1 + \cos \sqrt[4]{20}\, d_{12}z)^2 + 2(\sin \sqrt[4]{20}\, d_{12}z)^2\}$$

$$|E_2|^2 = \frac{3\sqrt{5}+5}{40}(1 - \cos \sqrt[4]{20}\, d_{12}z)^2 + \frac{1}{\sqrt{20}}(\sin \sqrt[4]{20}\, d_{12}z)^2$$

$$|E_3|^2 = \frac{3\sqrt{5}-5}{40}(1 - \cos \sqrt[4]{20}\, d_{12}z)^2 + \frac{1}{\sqrt{20}}(\sin \sqrt[4]{20}\, d_{12}z)^2. \tag{24}$$

Complete power transfer takes place when

$$\sqrt[4]{20}\, d_{12}z = (2k+1)\pi \; [k = 0, \pm 1, \pm 2 \cdots].$$

A feature of this solution (unlike others encountered) is that the variation of the amplitudes with distance is periodic.

Complete Power Transfer in Waveguides

Let us consider three coupled waveguides, where all the phase and energy velocities are in the same direction and waveguides 2 and 3 are not coupled.

A practical example might take the form of a power divider in which the power in 1 is transferred to particular modes in 2 and 3 in a predetermined ratio. In this example we assume that

$$\frac{\beta_2}{\beta_3} \quad \text{and} \quad \frac{|E_2(L)|^2}{|E_3(L)|^2} = s \tag{25}$$

are given, and β_1/β_3 and d_{13}/d_{12} are to be found.

It can be shown from (19), (20), and (25) that

$$s = -a. \tag{26}$$

Having obtained the value of a, the ratio d_{12}/d_{13} can be calculated from (19), or from Fig. 9. From the definitions of p and r [Eq. (11)] we get furthermore

$$\frac{\beta_1}{\beta_3} = \frac{1}{1-a}\left[1 - a\frac{\beta_2}{\beta_3}\right]. \tag{27}$$

Assuming for example, $\beta_2/\beta_3 = 0.8$ and $s = 0.5$ we get $\beta_1/\beta_3 = 0.93$ and $d_{12}/d_{13} = 0.89$. The power in the waveguides for the above values of the parameters is shown in Fig. 11 as a function of the normalized distance $d_{12}z$.

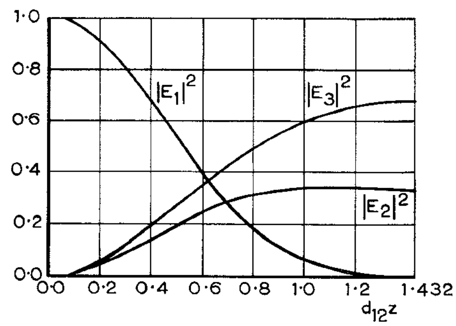

Fig. 11—The power in the waveguides as a function of the normalized distance, $d_{12}z$.

Double Beam Backward-Wave Oscillator

In a backward-wave oscillator utilizing two separate electron beams[11] so disposed that the interaction between them may be neglected, five waves are playing essential roles: the backward circuit wave, the two fast, and the two slow waves. However, if QC is large enough the problem can be greatly simplified. It is sufficient then to take account of the interaction of the backward circuit wave with the two slow waves.

Thus we can identify wave 1 with the backward circuit wave, and waves 2 and 3 with the slow waves. Since all the energy velocities are in the same direction, $f_{12} = f_{13} = 1$. The tube will oscillate if the power contained in the slow waves can be completely transferred to the backward circuit wave.

Let us investigate first the simplest case, when both beams (and thus both slow waves) are identical. Complete power transfer takes place if the propagation coefficient of the backward circuit wave is equal to those

[8] R. Kompfner, "On the operation of the traveling wave tube at low level," *J. Brit. IRE*, vol. 10, pp. 283–289; August–September, 1950.

[9] H. R. Johnson, "Kompfner dip conditions," PROC. IRE, vol. 43, p. 874; July, 1955.

[10] R. D. Weglein, "Backward wave oscillator starting conditions," IRE TRANS. ON ELECTRON DEVICES, vol. ED-4, pp. 177–179; April, 1957.

[11] E. A. Ash and A. C. Studd, "Multiple Beam Backward Wave Oscillators," presented at the Electron Tube Conf., Mexico City, Mexico; June, 1959.

of the slow waves, and the relation $(d_{12})_2L=\pi/2\sqrt{2}$ applies (15), where $(d_{12})_2$ is the coupling coefficient between the backward circuit wave and one of the slow waves at the start of oscillation. If there is only a single beam, the condition of start oscillation[7] is $(d_{12})_1L=\pi/2$, where $(d_{12})_1$ is the coupling coefficient between the backward circuit wave and the slow wave. Thus the necessary value of the coupling coefficient for the start of oscillation is smaller if both beams are present. Keeping the beam voltage constant, the ratio of the starting currents is as follows:

$$\frac{I_2}{I_1}=\left[\frac{(d_{12})_2}{(d_{12})_1}\right]^4=\frac{1}{4}. \tag{28}$$

Thus the beam current in a double-beam backward-wave oscillator drops by a factor 4, and the total current is still only half of that which is necessary in the single beam device.

If the beam voltages are slightly different, the coupling coefficients can still be regarded as identical because they are slowly varying functions of the beam voltage. Thus, applying the formulas of Section IV, the propagation coefficient of the backward circuit wave is the arithmetical mean of the propagation coefficients of the slow waves.

It may be seen from (17) that for finite voltage differences the starting current increases, which agrees qualitatively with the experimental results.[11] If

$$r^2 > 2d_{12}^2,$$

the amplitude of the backward circuit wave cannot be made zero. Thus, beyond a certain voltage difference, no oscillation can be obtained, however long the circuit.

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Noise Figures of Reflex Klystron Amplifiers*

KORYU ISHII†

Summary—**The noise figure of the 2K25 reflex klystron amplifier was investigated. The noise figure of the reflex klystron amplifier depends on operating frequency, electronic impedance, circuit impedance, and operating electronic mode. Experimental results show that a noise figure of 5 db is possible under particularly carefully adjusted conditions. In order to obtain the low-noise figure, careful electronic tuning and the impedance adjustments are particularly important. Generally, relatively low noise figures were obtained when the electronic tuning was good. Noise figures of cascaded reflex klystron amplifiers were also investigated experimentally. Noise figures of the cascaded amplifier were generally higher than that of the single stage amplifier, but still low enough to use this reflex klystron amplifiers as a preamplifier of a microwave receiver to increase the sensitivity of the receiving system.**

Introduction

The use, as regenerative or negative conductance amplifiers, of reflex klystrons originally designed for use in oscillators, would offer several advantages to microwave receiver design. Ordinary, small-power reflex klystrons are relatively inexpensive, and require neither the high voltages used in TW tubes nor the great magnetic force necessary in magnetrons.

There is some controversy about such an application for reflex klystrons. In the first place, it is questioned whether employment of the reflex klystron amplifier really does increase the sensitivity of a microwave receiver. To increase the receiver's sensitivity, the reflex klystron would have to provide a good gain and at the same time have a low noise figure.

Several papers have been published describing the gain achieved with reflex klystron amplifiers. Okabe[1] obtained a gain of over 20 db at 3000 mc with a 707B reflex klystron. Ishii[2,3] obtained a gain of more than 16 db at 9760 mc with a 723A/B reflex klystron. Quate, Kompfner and Chisholm[4] reported a gain of more than 30 db at 11,000 mc with a WE445A reflex klystron. These papers demonstrate that a substantial gain improvement is possible, but no useful data on noise figures was obtained. For example, Okabe reported a noise figure of less than 7 db but Quate reported 40 db. Clearly, a study of the noise figure itself was required if the value of the reflex klystron amplifier was to be verified or denied.

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† Dept. of Elec. Engrg., Marquette University, Milwaukee, Wis.

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